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Isgur-Wise Function From Bethe-Salpeter Amplitude

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ABSTRACT

We develop the improved ladder approximation to QCD in order to apply it to the heavy quark mesons. The resulting Bethe-Salpeter equation is expanded in powers of the inverse heavy quark mass $1/M$, and is shown to be consistent with the heavy quark spin symmetry. We calculate numerically the universal leading order BS amplitude for heavy pseudoscalar and vector mesons, and use this to evaluate the Isgur-Wise function and the decay constant F_B. The resulting Isgur-Wise function predicts a large charge radius, $\rho^2 = 1.8 - 2.0$, which when fitted to the ARGUS data corresponds to the value $V_{\rm cb} = .044 - .050$ for the Kobayashi-Maskawa matrix element.

1. Introduction

Recently there has been a great deal of interest in using B meson physics to test the detailed structure of the Standard Model and to reveal new physics. When we consider systems containing a heavy quark (such as the b quark and, possibly, the c quark) a new symmetry, the heavy quark spin-flavour symmetry[1] appears. This symmetry, which becomes exact in the limit when the heavy quark mass M goes to infinity, may be used to predict various properties of the heavy mesons; these have been systematically studied using the Heavy Quark Effective Theory.[2,3,4] For instance, when $M \to \infty$, every semi-leptonic form factor can be expressed in terms of a single universal function, the Isgur-Wise function.[2] At the moment there is a great deal of work being done to understand the semi-leptonic weak decay process $B \to D^{(*)} l\nu$ and thence to extract the Kobayashi-Maskawa matrix element V_{cb}:[5] it is therefore important to have a reliable functional form for the Isgur-Wise function. Unfortunately the heavy quark symmetry tells us nothing about the Isgur-Wise function away from the kinematical end point. In order to calculate it at any other point we have to know the details of the strong interaction physics of the light degrees of freedom in the heavy mesons.

In this paper we develop the improved ladder approximation to the QCD Bethe-Salpeter equation as an expansion in powers of $1/M$. We show that it is indeed consistent with the heavy quark symmetry, and apply the results to calculate the Isgur-Wise function. In previous papers[6,7] we have shown that the improved ladder approximation is capable of giving rather a good description of the physics of light mesons, and we therefore expect it to provide a reasonable approximation to the physics of the light degrees of freedom in heavy quark mesons.

We calculate the Isgur-Wise function as a one loop diagram containing two meson BS amplitudes and a single current insertion. The main problem which we must handle is the following: whilst the overall boundstate momenta must remain timelike, we have to Wick-rotate the loop momentum to evaluate the integral. Then the combinations of the loop momentum and the overall boundstate

momentum which appear in the arguments of the quark mass functions (and also, depending on the improvement scheme, in the argument of the running coupling constant) becomes complex (non-real). Moreover, when we calculate the Isgur-Wise function, we work in the rest frame of one of the two mesons but the other must be boosted to a finite velocity, and this corresponds to the BS amplitude with complex relative momentum. We are thus forced to calculate the BS amplitude with complex arguments. We propose a new method to do this analytic continuation using the BS equation itself. Because of this complication, the numerical work becomes quite hard — the Isgur-Wise function, for instance, appears as a five-dimensional integral.

In order to illustrate the main points of the method and obtain numerical results without being too much involved in such complications, we here ignore the running of the quark masses and work with a fixed value of the light quark mass m. We determine m by solving the equation $m = \Sigma(am^2)$ where Σ is the light-quark mass function determined from the ladder Schwinger-Dyson equation, and a is an unknown parameter. We have calculated the Isgur-Wise function for the two cases which we regard as typical, $a = 1$ and $a = 4$. A full treatment including the quark mass functions derived from the SD equation will be reserved for a later work. We actually find that the general shape of the Isgur-Wise function and the value of the charge radius ρ^2 are rather independent of the value of the fixed light-quark mass. Consequently we expect that the 'true' values obtained by including the mass function $\Sigma(x)$ will not differ significantly from our present results. The heavy meson decay constants, on the other hand, which we also calculate do not show this independence of the value of m, and so we have to wait for the full treatment.

The rest of this paper is organised as follows. In section 2 we present our formalism for treating the heavy meson BS equation as an expansion in powers of inverse heavy quark mass $1/M$, and we derive the leading order BS equation in a component form. An exact expression for the decay constant F_B in terms of the BS amplitude is given in section 3. In section 4 we explain how to calculate the Isgur-Wise function and show that our approximation is actually consistent with current

conservation in the equal heavy quark mass case. In section 5 we demonstrate that the leading order BS equation satisfies the heavy quark spin symmetry and discuss the implication for the form of BS amplitudes for pseudoscalar, vector, scalar and axial-vector mesons. Section 6 is devoted to the details of the numerical calculations. In section 6.1 we fix a suitable analytic functional form for the running coupling constant. In section 6.2 we use this coupling and solve the SD equation for the light quark mass function $\Sigma(x)$. We need Σ both in order to fix the light quark mass which we use in the BS equation, and also to fix the overall mass scale in MeV. We fix the scale by calculating the Pagels-Stokar pion decay constant and defining the result to $93\sqrt{2}$MeV. In section 6.3 we solve the BS equation and calculate the decay constant F_B and the energy eigenvalues of the ground state and the first excited state. Finally, in section 6.4, we present the results of our calculation of the Isgur-Wise function $\xi(t)$.

2. BS Amplitude and Notation

The BS amplitude of a meson boundstate $|B(\mathbf{q})\rangle$ of total momentum q containing a heavy quark Ψ and a light anti-quark $\bar{\psi}$ is defined by

$$\langle 0|\, \mathrm{T}\Psi_{i\alpha}(x)\bar{\psi}^j_\beta(y)\, |B(\mathbf{q})\rangle = e^{-iqX} \int \frac{d^4p}{(2\pi)^4}\, e^{-ipr}\ \delta^j_i\ \chi_{\alpha\beta}(p;q)\ , \tag{2.1}$$

where i, j and α, β denote colour and Dirac spinor indices respectively. The coordinate X^μ and the relative coordinate r^μ are defined by

$$X^\mu \equiv \zeta x^\mu + \eta y^\mu\ , \qquad r^\mu \equiv x^\mu - y^\mu\ .$$

where ζ and η are real parameters satisfying $\zeta + \eta = 1$. p is the therefore the relative momentum. Although in the relativistic case any value may in principle be chosen for ζ, we make the following 'natural' choice

$$\zeta = \frac{M}{M+m}\ , \qquad \eta = \frac{m}{M+m}\ ,$$

with M and m being the masses of the heavy and light quarks, respectively. With this choice, X^μ is the centre-of-mass coordinate of the system; it has the technical

advantage of making it legitimate to perform a Wick rotation in the BS equation.[8] For later use we here define the 4-velocity v^μ of the boundstate:

$$v^\mu = q^\mu / M_B \ , \qquad\qquad v^2 = 1 \ , \tag{2.2}$$

where M_B is the boundstate mass, $q^2 = M_B^2$.

It is convenient to introduce a ket notation to denote the bispinor BS amplitude:

$$\|\chi\rangle\!\rangle \quad \longleftrightarrow \quad \delta_i^j \, \chi_{\alpha\beta}(p;q) \ . \tag{2.3}$$

The conjugate BS amplitude which is denoted by a bra state is defined by

$$\langle\!\langle\chi\| \quad \longleftrightarrow \quad \delta_i^j \, \bar\chi_{\alpha\beta}(p;q) \equiv \delta_i^j \, \gamma_0 \, [\chi_{\alpha\beta}(p^*;q)]^\dagger \gamma_0 \ . \tag{2.4}$$

Note that the complex conjugate p^* of the argument p appears in the RHS: this is necessary for the general case where the relative momentum p becomes complex after Wick rotation. In fact, even when p is real, the $i\epsilon$ prescription in the Feynman propagator implies that the Minkowskian theory should be understood as that obtained by analytic continuation of the Euclidian theory. Considering the limit $\mathrm{Arg}\, p^0 \to 0+$, one can see that $\bar\chi$ defined this way does indeed coincide with the conventional conjugate BS amplitude:*

$$\langle B(\mathbf{q})| \, \mathrm{T}\psi_{j\beta}(y)\bar\Psi_\alpha^i(x) \, |0\rangle = e^{iqX} \int \frac{d^4p}{(2\pi)^4} \, e^{ipr} \ \delta_j^i \, \bar\chi_{\beta\alpha}(p;q) \ . \tag{2.5}$$

We also introduce a (non-positive-definite) inner product:

$$\langle\!\langle\psi\|\chi\rangle\!\rangle \equiv N_c \int \frac{d^4p}{i(2\pi)^4} \, \mathrm{tr}[\bar\psi(p;q)\chi(p;q)] \ , \tag{2.6}$$

where $N_c = 3$ comes from the trace over the colour index and the trace in the RHS is taken over the bispinor indices alone.

$\star$ One can also confirm this fact by seeing that the BS equation for the conventional conjugate BS amplitude is indeed satisfied by our conjugate amplitude (2.4).

The bispinor BS amplitude χ may be expanded into invariant component amplitudes. Here we are mainly interested in the pseudoscalar boundstate for which here there are four such amplitudes: in the present context of the heavy quark system, it is convenient to use the form

$$\chi(p;q) \ = \Lambda_+\big(A(u,x)+B(u,x)\not{\boldsymbol{p}}\big)\gamma_5+\Lambda_-\big(C(u,x)+D(u,x)\not{\boldsymbol{p}}\big)\gamma_5 \ , \tag{2.7}$$

where the projection operator $\Lambda_\pm$ and the 4-momentum $\boldsymbol{p}^\mu$, the spatial projection of p^μ, are defined by

$$\Lambda_\pm \equiv \frac{1\pm\not{v}}{2} \ , \qquad \boldsymbol{p}^\mu \equiv p^\mu-(p\cdot v)v^\mu \ ,$$

and the variables u and x are

$$iu=p\cdot v \ , \qquad x=\sqrt{-\boldsymbol{p}^2}=\sqrt{-p^2-u^2} \ . \tag{2.8}$$

The components of the conjugate BS amplitude $\bar{\chi}$ are defined similarly:

$$\bar{\chi}(p)=-\gamma_5\big(\bar{A}(u,x)+\bar{B}(u,x)\not{\boldsymbol{p}}\big)\Lambda_+-\gamma_5\big(\bar{C}(u,x)+\bar{D}(u,x)\not{\boldsymbol{p}}\big)\Lambda_- \ . \tag{2.9}$$

Note that iu is the time component p^0 of p^μ in the restframe of the boundstate where $v^\mu=(1,\mathbf{0})$. In this frame, u becomes real after Wick rotation, $p^\mu \ \rightarrow \ p_{\rm E}^\mu = (u,\mathbf{p})$, and then eq.(2.4) gives

$$\bar{\chi}(p)=-\gamma_5\big(A^*(-u,x)+B^*(-u,x)\not{\boldsymbol{p}}\big)\Lambda_+-\gamma_5\big(C^*(-u,x)+D^*(-u,x)\not{\boldsymbol{p}}\big)\Lambda_- \ . \tag{2.10}$$

In view of the BS equation (given shortly in (2.13)), we can easily see the relations $X^*(-u,x)$
$= X(u,x)$ for $X=A,B,C,D$, *provided that* the boundstate mass M_B, determined as a eigenvalue of the BS equation, is *real.* This is because when $q^\mu = M_B v^\mu$ is

real, in the Wick rotated form of the BS equation (in which the integration factor $\int d^4k/i(2\pi)^4$ becomes real), the imaginary quantity appearing is only through $p^0 = iu$ (or $k_0 \equiv iv$) so that the complex conjugation of the whole equation is equivalent to changing the sign of u (and v). So, for the true boundstate solutions corresponding to real mass eigenvalue M_B, we have simply

$$\begin{aligned} \bar{A}(u,x) &= A(u,x) \qquad \bar{B}(u,x) = B(u,x) \\ \bar{C}(u,x) &= C(u,x) \qquad \bar{D}(u,x) = D(u,x) \ . \end{aligned} \tag{2.11}$$

To simplify the discussion, we use the following abbreviation for momentum integration and for gamma matrix multiplication on the bispinor χ:

$$\begin{aligned} &\int_p \equiv \int \frac{d^4p}{i(2\pi)^4} \equiv \int \frac{d^4p_{\rm E}}{(2\pi)^4} \equiv \int \frac{x^2 dx\, du\, d\cos\theta}{8\pi^3} \ , \\ &\Gamma \otimes \Delta \ \|\chi\rangle\!\rangle \quad \longleftrightarrow \quad \delta_i^j \ (\Gamma\, \chi\, \Delta)_{\alpha\beta} \ . \end{aligned} \tag{2.12}$$

In this notation the BS equation becomes

$$S_{\rm H}(p+\zeta q) \otimes S_{\rm L}(p-\eta q)\, \|\chi\rangle\!\rangle = K \|\chi\rangle\!\rangle \tag{2.13}$$

where $S_{\rm H}$ and $S_{\rm L}$ are (i times) the inverse propagators of the heavy quark and the light quark respectively

$$S_{\rm H} \equiv \not p + \zeta \not q - M \ , \qquad S_{\rm L} \equiv \not p - \eta \not q - m \ , \tag{2.14}$$

and K stands for the Bethe-Salpeter kernel. In the improved ladder approximation K is given by the following one-gluon-exchange form:

$$\begin{aligned} K\|\chi\rangle\!\rangle &\equiv \int_k g^2 C_2\, D_{\mu\nu}(p-k) \gamma^\mu \chi(k) \gamma^\nu \ , \\ D_{\mu\nu}(k) &= \frac{g_{\mu\nu} - k_\mu k_\nu / k^2}{k^2} \ , \end{aligned} \tag{2.15}$$

where C_2 is the second Casimir given by $C_2 = (N_c^2 - 1)/2N_c$ and g^2 is a suitable running coupling constant. Since the running of the coupling is mainly caused by

the gluon self-energy corrections, probably the best choice of the argument of the running coupling constant is to use the gluon momentum and take $g^2\big((p_{\rm E}-k_{\rm E})^2\big)$, where $p_{\rm E}$ is Euclidean momentum; $p_{\rm E}^2=-p^2$. In this paper we take for simplicity $g^2\big(p_{\rm E}^2+k_{\rm E}^2\big)$. We have discussed this choice of argument before[9] and have shown that it does not differ from the ideal case very much. The precise functional form of our running coupling constant will be given later.

In the context of the heavy quark system, it is convenient to adopt the following boundstate normalisation

$$\big\langle B(\mathbf{q})|B(\mathbf{q}')\big\rangle=(2\pi)^3\,2v_0\,\delta^3(\mathbf{q}-\mathbf{q}')\ . \tag{2.16}$$

This differs from the usual invariant normalisation in that we have used v^0 in place of $q^0=M_Bv^0$: the usual invariantly normalised state is therefore given by $|B(\mathbf{q})\rangle\sqrt{M_B}$. By considering the contribution of the intermediate boundstate to the two heavy two light quark Greens function in the standard way, we can show that our BS amplitude may be normalised by requiring[8]

$$\zeta\langle\!\langle\chi\|\gamma_\mu\otimes S_{\rm L}\|\chi\rangle\!\rangle-\eta\langle\!\langle\chi\|S_{\rm H}\otimes\gamma_\mu\|\chi\rangle\!\rangle=2v_\mu\ . \tag{2.17}$$

The BS equation gives an eigenvalue equation for the boundstate mass M_B or, equivalently, for the binding energy E defined as

$$M_B=M+m-E=\zeta^{-1}(M-\zeta E)\ . \tag{2.18}$$

We now expand all quantities as power series in m/M. The light quark mass m (which is the constituent quark mass and comes almost entirely from dynamical chiral symmetry breaking) is of order $\Lambda_{\rm QCD}$, and we therefore regard $\Lambda_{\rm QCD}$ as $O(1)$ in this expansion. Thanks to our choice of ζ and η, the relative momenta p and

k are also of the order of $\Lambda_{\rm QCD}$, and therefore also $O(1)$. The binding energy and BS amplitude are expanded as

$$\begin{aligned}
\zeta E &= E_0 + (\frac{m}{M})E_1 + (\frac{m}{M})^2 E_2 + \cdots \ , \\
\chi &= \chi_0 + (\frac{m}{M})\chi_1 + (\frac{m}{M})^2 \chi_2 + \cdots \ ,
\end{aligned} \tag{2.19}$$

with corresponding invariant amplitudes:

$$\begin{aligned}
\chi_0(p) &= \Lambda_+\big(A_0(u,x) + B_0(u,x)\not{\boldsymbol{p}}\big)\gamma_5 + \Lambda_-\big(C_0(u,x) + D_0(u,x)\not{\boldsymbol{p}}\big)\gamma_5 \ , \\
\chi_1(p) &= \Lambda_+\big(A_1(u,x) + B_1(u,x)\not{\boldsymbol{p}}\big)\gamma_5 + \Lambda_-\big(C_1(u,x) + D_1(u,x)\not{\boldsymbol{p}}\big)\gamma_5 \ ,
\end{aligned} \tag{2.20}$$

and so on. The expansion of the quark inverse propagators is (notice that $S_{\rm H}$ starts with a -1-st order term)

$$\begin{aligned}
S_{\rm H} &= (\frac{m}{M})^{-1} S_{\rm H-1} + S_{\rm H0} + (\frac{m}{M}) S_{\rm H1} + \cdots \ , \\
&\quad S_{\rm H-1} \equiv m(\not{v} - 1) = -2m\Lambda_- \ , \\
&\quad S_{\rm H0} \equiv \not{p} - E_0\not{v} \ , \\
&\quad S_{\rm H1} \equiv -E_1\not{v} \ , \\
S_{\rm L} &= S_{\rm L0} + (\frac{m}{M}) S_{\rm L1} + (\frac{m}{M})^2 S_{\rm L2} + \cdots \ , \\
&\quad S_{\rm L0} \equiv \not{p} - m(1+\not{v}) = \not{p} - 2m\Lambda_+ \ , \\
&\quad S_{\rm L1} \equiv E_0\not{v} \ , \\
&\quad S_{\rm L2} \equiv E_1\not{v} \ .
\end{aligned} \tag{2.21}$$

Substituting these expansions into the BS equation (2.13), we find that at -1-st order

$$S_{\rm H-1} \otimes S_{\rm L0} \| \chi_0 \rangle\!\rangle = 0 \ , \tag{2.22}$$

at zeroth order

$$S_{\rm H-1} \otimes \Big(S_{\rm L0}\|\chi_1\rangle\!\rangle + S_{\rm L1}\|\chi_0\rangle\!\rangle\Big) + S_{\rm H0} \otimes S_{\rm L0}\|\chi_0\rangle\!\rangle = K\|\chi_0\rangle\!\rangle \ , \tag{2.23}$$

and so on. Since $S_{\rm H-1} \propto \Lambda_-$, the -1-st order equation (2.22) simply implies that $\chi_0 \propto \Lambda_+$ so that we find $C_0(u,x) = D_0(u,x) = 0$. This then causes the

term $S_{\mathrm{H}-1} \otimes S_{\mathrm{L}1} \| \chi_0 \rangle\!\rangle$ in the zeroth order equation to vanish. Moreover in the zeroth order equation, the first term, $S_{\mathrm{H}-1} \otimes S_{\mathrm{L}0} \| \chi_1 \rangle\!\rangle$, contributes only to the piece proportional to Λ_-, so that we have

$$\Lambda_+ \left[S_{\mathrm{H}0} \otimes S_{\mathrm{L}0} - K \right] \| \chi_0 \rangle\!\rangle = 0 \ . \tag{2.24}$$

Projecting out the Λ_+ pieces by taking $\frac{1}{2}\,\mathrm{tr}[\gamma_5 \Lambda_+ \times (2.24)]$ and $\frac{1}{2}\,\mathrm{tr}[\gamma_5 \not{\boldsymbol{p}} \Lambda_+ \times (2.24)]$, the zeroth (or leading) order BS equation (2.24) becomes

$$(E_0 - iu)\mathcal{M} \begin{pmatrix} A_0(p) \\ B_0(p) \end{pmatrix} = \int \frac{y^2 dy dv}{4\pi^3}\, g^2 C_2\, K_0(p,k) \begin{pmatrix} A_0(k) \\ B_0(k) \end{pmatrix} \ , \tag{2.25}$$

where the arguments p and k stand for (u, x) and (v, y) respectively, $\mathcal{M}$ is a 'metric' matrix given by

$$\mathcal{M} \equiv \begin{pmatrix} iu & -x^2 \\ -x^2 & x^2(iu - 2m) \end{pmatrix} \ , \tag{2.26}$$

and the kernel $K_0(p,k)$ is obtained after integrating K over the angular variable $\cos\theta = \mathbf{k} \cdot \mathbf{p} / |\mathbf{k}||\mathbf{p}|$. The explicit matrix expression for $K_0(p,k)$ is given in the Appendix.

In the leading order of the m/M expansion the normalisation condition (2.17) becomes

$$\frac{1}{2} \langle\!\langle \chi_0 \| \not{v} \otimes S_{\mathrm{L}0} \| \chi_0 \rangle\!\rangle = 1 \ , \tag{2.27}$$

which when rewritten in terms of the invariant amplitudes gives

$$N_c \int \frac{x^2 dx du}{4\pi^3} \left\{ \begin{pmatrix} \bar{A}_0(p) & \bar{B}_0(p) \end{pmatrix} \mathcal{M} \begin{pmatrix} A_0(p) \\ B_0(p) \end{pmatrix} \right\} = 1 \tag{2.28}$$

with the metric matrix $\mathcal{M}$ defined in eq.(2.26).

The BS equation (2.25) is a linear eigenvalue equation $\mathcal{A}\,\|\chi\rangle\!\rangle = E_0\mathcal{M}\,\|\chi\rangle\!\rangle$ with $\mathcal{A} = g^2C_2K_0 + iu\mathcal{M}$ and metric $\mathcal{M}$. Further, both $\mathcal{A}$ and the metric $\mathcal{M}$ are hermitian with respect to the inner product $\langle\!\langle\psi\|\chi\rangle\!\rangle$ defined in (2.6): $[\mathcal{A}_{i,j}(-u,x;-v,y)]^* = \mathcal{A}_{j,i}(v,y;u,x)$ $(i,j=1,2)$ and so on. As suggested by the normalisation condition (2.28), the natural inner product to this BS system is given by $\langle\!\langle\psi\|\mathcal{M}\|\chi\rangle\!\rangle$ and the norm by $\|\chi\|^2 = \langle\!\langle\chi\|\mathcal{M}\|\chi\rangle\!\rangle$ accordingly. Indeed, the BS equation (2.25), with the help of the hermiticity of $\mathcal{A}$ and $\mathcal{M}$, leads to an equality $(E_0^{\psi *} - E_0^{\chi})\langle\!\langle\psi\|\mathcal{M}\|\chi\rangle\!\rangle = 0$ for arbitrary two eigenstates $\|\chi\rangle\!\rangle$ and $\|\psi\rangle\!\rangle$ belonging to eigenvalues E_0^{χ} and E_0^{ψ},respectively. This implies, in particular, that the energy eigenvalues E_0 must be *real* as far as the eigenstate has non-zero norm $\langle\!\langle\chi\|\mathcal{M}\|\chi\rangle\!\rangle \neq 0$. Namely all solutions to our BS equation (2.25) for which the LHS of Eq.(2.28) is non-zero correspond to real eigenvalues E_0. In numerical work we actually find many solutions to (2.25) with complex eigenvalues E_0, but they all turn out to have zero-norm in accord with this observation.

3. Decay Constant

The pseudoscalar meson decay constant F_B is defined by

$$iF_B q_\mu e^{-iqx} \equiv \langle 0|\,\bar{\psi}(x)\gamma_\mu\gamma_5\Psi(x)\,|B(\mathbf{q})\rangle\,\sqrt{M_B}\ , \tag{3.1}$$

where the factor of $\sqrt{M_B}$ is included so that the state $|B(\mathbf{q})\rangle\,\sqrt{M_B}$ satisfies the usual relativistic normalisation. Our definition would correspond to a pion decay constant $F_\pi = 93\times\sqrt{2}$ MeV.

Setting $r \equiv x - y = 0$ in the definition of the BS amplitude (2.1) we find

$$F_B\, q_\mu = -N_c \int_p \mathrm{tr}[\gamma_\mu\gamma_5\chi(p;q)] \times \sqrt{M_B}\ . \tag{3.2}$$

Substituting the component expression of the BS amplitude χ, we have

$$F_B\sqrt{M_B} = N_c \int \frac{x^2 dx du}{2\pi^3}\,[\ A(u,x) - C(u,x)\ ]\ , \tag{3.3}$$

which is an exact formula.

4. Isgur-Wise Function

The expression for a heavy quark current operator $\bar{\Psi}'\gamma_\mu\Psi$ between two pseudoscalar boundstates B' and B containing heavy quarks Ψ' and Ψ with masses M' and M may be described in terms of two form factors $f(t)$ and $g(t)$ as

$$\left\langle B'(\mathbf{q}')\right| \bar{\Psi}'(x)\gamma_\mu\Psi(x)\left|B(\mathbf{q})\right\rangle \equiv \left[(v+v')_\mu f(t) + (v-v')_\mu g(t)\right] e^{i(q'-q)x} \tag{4.1}$$

where $t = v \cdot v'$. When the initial and final mesons are the same (that is $\Psi = \Psi'$ and $M_B = M_{B'}$), current conservation leads to $g(t) = 0$. The Isgur-Wise function $\xi(t)$ [2] is defined as the leading term in the expansion of the first form factor $f(t)$ in powers of m/M (or m/M')

$$\begin{aligned} f(t) &= \xi(t) + \big(\frac{m}{M}\big) f_1(t) + \big(\frac{m}{M}\big)^2 f_2(t) + \cdots\ , \\ g(t) &= \big(\frac{m}{M}\big) g_1(t) + \big(\frac{m}{M}\big)^2 g_2(t) + \cdots\ . \end{aligned} \tag{4.2}$$

The absence of the leading order term $g_0(t)$ in $g(t)$ can easily be seen : using the equality $\Lambda_+(\not{v}' - \not{v})\Lambda_+ = 0$, we find $\bar{\chi}_0(p';q')(\not{v}' - \not{v})\chi_0(p;q) = 0$ (remember $\chi_0 = \Lambda_+\chi_0$ and $\bar{\chi}_0 = \bar{\chi}_0\Lambda_+$).

$\xi(t)$ is independent of the heavy quark masses M and M' and is universal over pseudoscalar and vector heavy-light quarkonia: exactly the same function appears in the expansion of the form factors in the vector meson case. In the present approximation using constant mass for the heavy quark, the vector current matrix element (4.1) may be calculated from the diagram of figure 1.

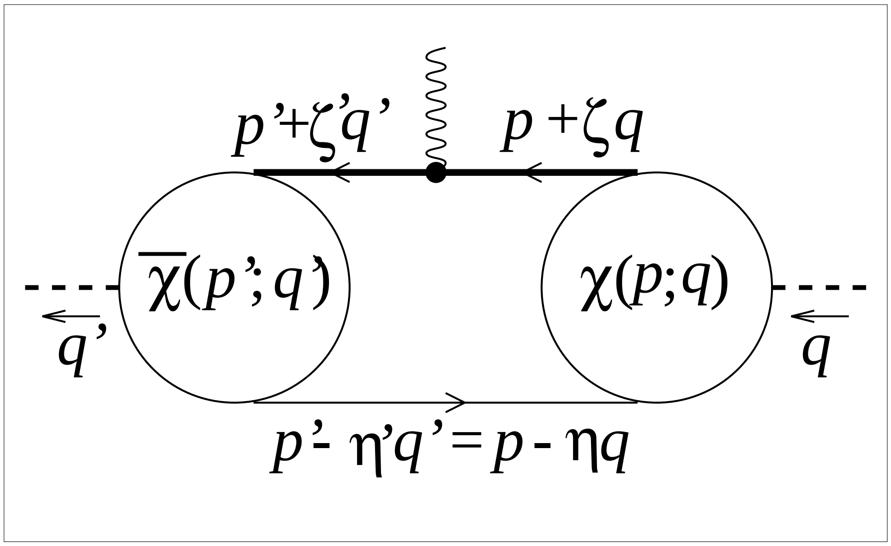

Fig.1 Feynman diagram used to calculate the Isgur-Wise function.

Thus we use

$$\begin{aligned}
&\left\langle B'(\mathbf{q}')\right| \bar{\Psi}'(0)\gamma_\mu \Psi(0)\left|B(\mathbf{q})\right\rangle \\
&\qquad = N_c \int \frac{d^4 p'_{\mathrm{E}}}{(2\pi)^4}\,\mathrm{tr}\left[\bar{\chi}(p';q')\gamma_\mu \chi(p;q) S_{\mathrm{L}}(p-\eta q = p'-\eta' q')\right]\ ,
\end{aligned} \tag{4.3}$$

where $\eta' \equiv m/(M'+m)$ and $\zeta' \equiv M'/(M'+m)$ for the final state meson B'. Actually, in the equal mass case, the approximation of eq.(4.3) is consistent with current conservation to all orders: using $\eta+\zeta = \eta'+\zeta' = 1$, $p-p' = \eta q - \eta' q'$ and $M = M'$ we have

$$\begin{aligned}
q - q' &= (\eta+\zeta)q - (\eta'+\zeta')q' = (p+\zeta q) - (p'+\zeta' q') \\
\longrightarrow\ \not{q} - \not{q}' &= (\not{p} + \zeta\not{q} - M) - (\not{p}' + \zeta'\not{q}' - M) = S_{\mathrm{H}} - S'_{\mathrm{H}}\ .
\end{aligned} \tag{4.4}$$

Using this equation and the BS equation (2.13), it is easy to show that $g(t) = 0$:

$$\begin{aligned}
M_B(v-v')^2 g(t) &= (q-q')^\mu \langle\!\langle \chi' \| \gamma_\mu \otimes S_{\mathrm{L}} \| \chi \rangle\!\rangle \\
&= \langle\!\langle \chi' \| (\not{q} - \not{q}') \otimes S_{\mathrm{L}} \| \chi \rangle\!\rangle \\
&= \langle\!\langle \chi' \| \big(S_{\mathrm{H}} \otimes S_{\mathrm{L}} - S'_{\mathrm{H}} \otimes S'_{\mathrm{L}}\big) \| \chi \rangle\!\rangle \qquad (\text{since } S_{\mathrm{L}} = S'_{\mathrm{L}}) \\
&= \langle\!\langle \chi' \| (\ K\ -\ K\ ) \| \chi \rangle\!\rangle = 0\ .
\end{aligned} \tag{4.5}$$

We make one small comment: it is known that if a running heavy quark mass

function determined from the ladder SD equation were used, the matrix element (4.1) should be calculated by including gluon ladders also in the vector current channel, since otherwise current conservation would be violated. However diagrams containing one or more gluon exchanges in this channel are easily seen to be suppressed by at least a power of $\left(\frac{m}{M}\right)^2$ and so the approximation we use satisfies current conservation to at least leading order in the m/M expansion. In fact $g_0(t) = 0$ as we have seen before.

The Isgur-Wise function $\xi(t)$ is easily obtained by taking the leading order term of eq.(4.3) and contracting with $(v + v')^\mu$: we find

$$2(1+t)\xi(t) = N_c \int_{p'} \mathrm{tr}\left[\bar{\chi}_0(p';q')\,(\not{v} + \not{v}')\chi_0(p;q)\,S_{\mathrm{L}0}(p-\eta q = p' - \eta' q')\right] . \quad (4.6)$$

After taking the trace, we find the expression in terms of the invariant amplitudes:

$$\xi(t) = \frac{N_c}{1+t}\int \frac{du'\,x'^2 dx' d\cos\theta}{8\pi^2}\left\{\begin{pmatrix}\bar{A}_0' & \bar{B}_0'\end{pmatrix}\begin{pmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}\begin{pmatrix}A_0\\ B_0\end{pmatrix}\right\} , \quad (4.7)$$

where $A_0' \equiv A_0(u', x')$, $A_0 \equiv A_0(u, x)$ and so on, and the elements of the matrix L are given by

$$\begin{aligned}
L_{11} &= k_1 + iu'(t+1)\\
L_{12} &= -k_1^2 - iu'k_1(2t+1) + (t+1)\left[mk_1 + iu'(t-1)(m-iu') - {x'}^2\right]\\
L_{21} &= k_1(2m - iu') - {x'}^2(t+1)\\
L_{22} &= -k_1^2(2m-iu') + k_1 t{x'}^2\\
&\qquad + (t+1)\left[k_1(iu'-m)(iu'-2m) + iu'{x'}^2 t - m{x'}^2(t+1)\right]
\end{aligned} \quad (4.8)$$

with

$$k_1 \equiv p' \cdot (v - v't) = p_1'\sqrt{t^2-1} . \quad (4.9)$$

We should note that the important condition $\xi(t{=}1) = 1$ is satisfied in our formu-

lation: actually, Eq.(4.6) may be rewritten into the form

$$\xi(t) = \frac{1}{1+t}\langle\!\langle \chi_0' \| \frac{\not{v} + \not{v}'}{2} \otimes S_{\rm L0} \| \chi_0 \rangle\!\rangle \ .$$

where t becomes 1, v' and χ_0' become v and χ_0, respectively, so that the RHS of this clearly reduces to the normalisation condition (2.27) and gives 1.

We evaluate the integral in (4.7) in the B' restframe in which $v' = (1, \mathbf{0})$. In this frame we label the components of p' as

$$\begin{aligned} & p' = (iu', p_1', p_2', p_3') = (iu', p_1', \mathbf{p}_\perp') \ , \\ & p_1' = x' \cos\theta \ , \qquad |\mathbf{p}_\perp'| = x' \sin\theta \ , \end{aligned} \tag{4.10}$$

and take the integration contour of $\int d^4 p_{\rm E}'$ over the real axis of u' and p_i'. The momentum p of the initial state BS amplitude $\chi(p;q)$ is determined by the relation

$$p - \eta q = p' - \eta' q' \quad \rightarrow \quad p = p' - \eta' q' + \eta q \ . \tag{4.11}$$

and becomes complex. We require the corresponding arguments u and x for the component amplitudes A_0 and B_0. Since $u = p \cdot v / i$ and $x = \sqrt{-p^2 + (p \cdot v)^2}$ are Lorentz invariants, it is simplest to calculate them in the v-restframe where $v = (1, \mathbf{0})$. In this frame the vectors v' and p' take the form (since $v \cdot v' = t$):

$$\begin{aligned} & v' = (t, \ \sqrt{t^2 - 1}, \ 0, \ 0) \ , \\ & p' = (iu't + p_1'\sqrt{t^2-1}, \ \ iu'\sqrt{t^2-1} + p_1' t, \ \ \mathbf{p}_\perp') \ , \end{aligned} \tag{4.12}$$

so that p is given by

$$\begin{aligned} p &= p' - \eta' q' + \eta q = p' - mv' + mv + O\big(\frac{m}{M}\big) \\ &= \Big(iu't + p_1'\sqrt{t^2-1} - m(t-1), \ \ (iu' - m)\sqrt{t^2-1} + p_1' t, \ \ \mathbf{p}_\perp'\Big) \ + \ O\big(\frac{m}{M}\big) \ . \end{aligned} \tag{4.13}$$

Therefore to leading order in m/M the arguments u and x of the initial state BS

amplitude A_0 and B_0 are given by

$$
\begin{aligned}
u &= \frac{1}{i} p \cdot v = u' t + i[m(t-1) - p'_1 \sqrt{t^2-1}] \ , \\
x &= \sqrt{\mathbf{p}^2} = \sqrt{\left[iu'\sqrt{t^2-1} + (p'_1 t - m\sqrt{t^2-1})\right]^2 + \mathbf{p}'^{\,2}_{\perp}} \ .
\end{aligned}
\tag{4.14}
$$

Note that u and x in (4.14) are complex. In order to find the BS amplitudes $A_0(u,x)$ and $B_0(u,x)$ at these complex values, we need to perform an analytic continuation. Fortunately this continuation can be done using the BS equation itself: we simply put the complex values u and x into the BS equation (2.25) and find $A_0(u,x)$ and $B_0(u,x)$ by performing the RHS integration over v and y, which requires only the knowledge of the BS amplitude with real variables v and y.

5. Heavy Quark Spin Symmetry of Leading Order BS Amplitudes

The full BS equation eq.(2.13) of course applies to any heavy quark light anti-quark boundstate $|B(\mathbf{q})\rangle$. So the -1st order equation eq.(2.22) tells us that the leading order BS amplitude χ_0 satisfies $\Lambda_-\chi_0 = 0$ for all boundstates (not just the pseudoscalar); $\chi_0 \propto \Lambda_+$. Therefore, for instance, the leading order BS amplitudes of pseudoscalar and vector boundstates generally have the following forms:

$$
\begin{aligned}
\chi_{0\mathrm{Ps}} &= \Lambda_+ \big(A + B\not{\boldsymbol{p}}\big)\gamma_5 \ , \\
\chi_{0\mathrm{V}} &= \Lambda_+ \big[\not{\epsilon}(A_V - B_V\not{\boldsymbol{p}}) + \epsilon \cdot p (C_V + D_V \not{\boldsymbol{p}})\big] \ ,
\end{aligned}
\tag{5.1}
$$

where ϵ^μ is the polarisation vector of the vector boundstate.

Let us now show that our (improved) ladder BS equation satisfies the heavy quark spin symmetry[2] in the leading order of m/M expansion, and that the pseudoscalar and vector BS amplitudes are given by

$$
\begin{aligned}
\chi_{0\mathrm{Ps}} &= \Lambda_+ \gamma_5 \big(A - B\not{\boldsymbol{p}}\big) \ , \\
\chi_{0\mathrm{V}} &= \Lambda_+ \not{\epsilon} \big(A - B\not{\boldsymbol{p}}\big) \ ,
\end{aligned}
\tag{5.2}
$$

with common functions A and B, and belong to a common energy eigenvalue. To

see this we rewrite the leading order BS equation eq.(2.24) in the form

$$(p \cdot v - E_0)\chi_0(p)(\not{p} - 2m\Lambda_+) = \int_k v^\mu \chi_0(k)\gamma^\nu \, g^2 C_2 \, D_{\mu\nu}(p-k) \tag{5.3}$$

with χ_0 subject to the constraint $\Lambda_- \chi_0 = 0$. The important point is that in eq.(5.3) the heavy quark spinor index α of the amplitude $\chi_{0\alpha\beta}$ (the left-leg) is left intact: there are no γ-matrix factors multiplying χ_0 from the left. The leading order BS equation is therefore invariant under any heavy quark spin rotation which commutes with Λ_- — commutativity with Λ_- is required to maintain the constraint $\Lambda_- \chi_0 = 0$. In particular, in the restframe of the boundstate in which $\Lambda_- = \frac{1-\gamma_0}{2}$, it is invariant under

$$\begin{aligned} \chi_0(p;q) &\to U(\boldsymbol{\theta})\chi_0(p;q) \ , \\ U(\boldsymbol{\theta}) &= \exp\left(i\boldsymbol{\theta} \cdot \frac{\boldsymbol{\sigma}}{2}\right) \ . \end{aligned} \tag{5.4}$$

In this frame, the polarisation vector ϵ^μ of the vector boundstate is of the form $\epsilon^\mu = (0, \boldsymbol{\epsilon})$ ($|\boldsymbol{\epsilon}| = 1$) since $\epsilon \cdot v \propto \epsilon \cdot q = 0$. Let us rotate the pseudoscalar amplitude through $\theta = \pi$ around the $\boldsymbol{\epsilon}$-axis by multiplying by $U(\pi\boldsymbol{\epsilon})$. Noting the relations

$$U(\pi\boldsymbol{\epsilon}) = i\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \ \text{ and } \ \Lambda_+ \boldsymbol{\sigma}\gamma_5 = \Lambda_+ \boldsymbol{\gamma} \ ,$$

we find

$$U(\pi\boldsymbol{\epsilon}) \ \Lambda_+ \gamma_5 \big(A - B\not{\boldsymbol{p}}\big) = i\Lambda_+ \boldsymbol{\epsilon} \cdot \boldsymbol{\gamma}\big(A - B\not{\boldsymbol{p}}\big) \ . \tag{5.5}$$

But (aside from a constant overall factor $-i$) the RHS is nothing but the vector meson BS amplitude $\chi_{0\mathrm{V}}$ as given in eq.(5.2). So if the pseudoscalar BS amplitude $\chi_{0\mathrm{Ps}}$ in eq.(5.2) satisfies the BS equation (5.3), then so does the vector amplitude $\chi_{0\mathrm{V}}$ in (5.2), and the energy eigenvalues are the same.

It is also important to note that the leading order BS equation (5.3) is totally independent of the heavy quark mass. This implies that the excitation energies determined as energy eigenvalues of that equation are also universal; namely, sufficiently heavy quark mesons with the same spin-parity or their partners under heavy quark spin symmetry should show almost the same excitation energy spectra.

In the same way as in the pseudoscalar and vector case, we can see that the scalar and axial vector mesons form a degenerate multiplet under the heavy quark spin symmetry, and that their leading order BS amplitudes are given in the form.

$$
\begin{aligned}
\chi_{0\mathrm{S}} &= \Lambda_+\big(A_S - B_S \not{p}\big) \ , \\
\chi_{0\mathrm{A}} &= \Lambda_+ \not{\epsilon} \big(A_S + B_S \not{p}\big)\gamma_5 \ .
\end{aligned}
\tag{5.6}
$$

6. Numerical Calculation

6.1. Choice of Running Coupling Constant

Asymptotic freedom requires that at scales $\mu^2 \gg \Lambda^2_{\mathrm{QCD}}$ the running coupling behaves (in the one loop approximation) as

$$
\alpha(\mu^2) = \frac{\alpha_0}{\ln(\mu^2/\Lambda^2_{\mathrm{QCD}})} \ , \tag{6.1}
$$

where the constant α_0 depends on the number of light quark flavours — we work with three light flavours for which $\alpha_0 = 4\pi/9$. Unfortunately we have no guidance on the form of the coupling outside this deep Euclidean region. Previous investigations of the improved ladder approximation [6,10] have simply continued the form of (6.1) so as to have no singularities on the real axis in the spacelike region $\mu^2 > 0$. However the prescriptions which were used are inadequate for the present calculation of the Isgur-Wise function. This is because argument $p^2_{\mathrm{E}} + k^2_{\mathrm{E}}$ of $\alpha(p^2_{\mathrm{E}} + k^2_{\mathrm{E}})$ becomes complex when we use the BS equation (2.25) to calculate $\chi(u, x)$ at complex argument. We thus require that the coupling be analytic and that it have no singularities within the integration region used to perform the analytic continuation for the boosted BS amplitude.

We adopt the following form:

$$\begin{aligned}\alpha(\mu^2) &= \frac{\alpha_0}{\ln(f(\mu^2/\Lambda_{\rm qcd}^2))} \ , \\ f(x) &= x + \kappa \ \ln\left[\ 1 + \exp\left(\frac{x_0 - x}{\kappa}\right)\ \right] \ ,\end{aligned} \tag{6.2}$$

with suitable real $\kappa > 0$ and $x_0 > 1$. $\Lambda_{\rm qcd}$ is a constant analogous to $\Lambda_{\rm QCD}$ which sets the scale of the coupling. Since we determine the value of $\Lambda_{\rm qcd}$ mainly from the infra-red behaviour of the coupling, we should not expect it to agree with $\Lambda_{\rm QCD}$ which is usually fixed using data at much higher energies. Notice that $f(x) \sim x$ for large positive x, so that the leading asymptotic behaviour of the coupling is correct regardless of the value of $\Lambda_{\rm qcd}$.

From eq.(6.2) one can show that α tends to the constant value $\alpha_{\rm max} = \alpha_0/\ln x_0$ in the low energy (or timelike) region $x \ll x_0$. We use the values

$$x_0 = 1.01,\ 1.05 \text{ and } 1.10,$$

corresponding to $\alpha_{\rm max} = 140.3,\ 28.6$ and 14.6 respectively.

Next we determine κ. $f(x)$ has branch point singularities at

$$x_{\rm s} \equiv x_0 + i\pi\kappa(2n+1) \ , \quad n \in \mathbf{Z} \ , \tag{6.3}$$

with the branch cuts extending horizontally to the left if the principal value of the logarithm is taken; these singularities give rise to corresponding singularities in $\alpha(\mu^2)$. So κ must be sufficiently large that these singularities and branch cuts lie outside the integration region used to calculate the BS amplitude $\chi(u,x)$ with complex arguments u and x. From eq.(4.13), the complex momentum squared $p_{\rm E}^2$ is given by

$$p_{\rm E}^2 = u'^2 - 2im(\,t-1\,)u' + x'^2 - 2mx'\cos\theta\sqrt{t^2-1} + 2m^2(\,t-1\,) \ , \tag{6.4}$$

where u' and x' run over real values. We can avoid the singularity in $f\big((p_{\rm E}^2 +$

$k_{\mathrm{E}}^2)/\Lambda_{\mathrm{qcd}}^2\big)$ if, when $\mathrm{Re}\,(\,p_{\mathrm{E}}^2 + k_{\mathrm{E}}^2\,)/\Lambda_{\mathrm{qcd}}^2 = x_0$,

$$|\mathrm{Im}\,(\,p_{\mathrm{E}}^2 + k_{\mathrm{E}}^2\,)| = 2m(t-1)u' < \pi\kappa\Lambda_{\mathrm{qcd}}^2\ , \ \text{for all} \quad u'\,,x'\,,k_{\mathrm{E}}^2\ >\ 0\ . \tag{6.5}$$

When $\mathrm{Re}\,(\,p_{\mathrm{E}}^2 + k_{\mathrm{E}}^2\,)/\Lambda_{\mathrm{qcd}}^2 = x_0$,

$$\begin{aligned} x_0\Lambda_{\mathrm{qcd}}^2 &= \mathrm{Re}\,(p_{\mathrm{E}}^2 + k_{\mathrm{E}}^2) \\ &= k_{\mathrm{E}}^2 + u'^2 + (\,x' - m\sqrt{\,t^2-1}\cos\theta\,)^2 + m^2(\,2t-2\ \ -(t^2-1)\cos^2\theta\,) \\ &\geq u'^2 - m^2(t-1)^2\ , \end{aligned} \tag{6.6}$$

so that

$$u' \leq u'_{\max} \equiv \sqrt{\,x_0\Lambda_{\mathrm{qcd}}^2 + m^2(\,t-1\,)^2\,}\ . \tag{6.7}$$

Thus we require

$$\kappa > \frac{2(t-1)}{\pi}\frac{m}{\Lambda_{\mathrm{qcd}}}\sqrt{x_0 + \frac{m^2}{\Lambda_{\mathrm{qcd}}^2}(\,t-1\,)^2} \tag{6.8}$$

for the range of values of t we consider. As will be seen below, at fixed κ, m/Λ_{qcd} is determined by x_0 and the RHS is a function of only x_0 and t. Using the data for m/Λ_{qcd} obtained below for the three values of x_0, we can estimate the RHS of (6.8) and find that $\kappa = 0.3$ allows us to calculate the Isgur-Wise function for $t < 1.55$. As will be seen below, this region of t covers the kinematically allowed region for the semi-leptonic B-decay which we are interested in. So we fix κ equal to 0.3 henceforth. The reader may have noticed that there are also singularities in eq.(6.2) when $f(x) = 0$ or $f(x) = 1$; it is easy to see that the condition (6.8) given above is sufficient to avoid these singularities.

6.2. Fixing the Mass Scale and the Light Quark Masses

In order to treat dynamical chiral symmetry breaking consistently in the ladder approximation we should use the dynamical quark mass function obtained from the improved ladder approximation to the SD equation, and we should use the same kernel in both the SD and BS equations. It is easy to see that this approach correctly leads to massless pions in the chiral limit.[6]

In fact, the inclusion of the running mass is plagued with a number of technical difficulties related to the consistency of the axial WT identity.[11] In the present calculation, there is the further problem that, after Wick rotation of the loop momentum, the momenta flowing through the light and heavy fermion propagators, $S_{\rm H}(p+\zeta q)$ and $S_{\rm L}(p-\eta q)$, become complex necessitating values of the quark mass function $\Sigma(x)$ with complex arguments $x=-(p+\zeta q)^2,\ -(p-\eta q)^2$. These are not insurmountable problems (in particular the quark mass function with complex argument may be evaluated using the SD equation in the same way as we calculated the boosted BS amplitude in Sect.3), however they considerably complicate the formalism. In this paper we therefore use fixed quark masses and defer the discussion using the running mass function to a future paper.

It turns out that the leading order calculations need no particular value of the heavy quark mass M, but we do have to fix the light quark mass m as well as the scale $\Lambda_{\rm qcd}$ used in the coupling. In order to do this we adopt the following procedure. First we solve numerically the improved ladder SD equation for the light quark mass function $\Sigma(p_{\rm E}^2)$, using the running coupling $\alpha(p_{\rm E}^2+k_{\rm E}^2)$ given in eq.(6.2);

$$\Sigma(x)=\frac{3C_2}{4\pi}\int_0^\infty dy\ \frac{\alpha(x+y)}{\max(x,y)}\ \frac{y\Sigma(y)}{y+\Sigma(y)^2}\ . \tag{6.9}$$

There is no unique definition of the constituent quark mass in terms of the quark mass function Σ; we work with the following two definitions[12]

$$\begin{aligned} m&=\Sigma(m^2)\quad \text{type I}\ ,\\ m&=\Sigma(4m^2)\quad \text{type II}\ . \end{aligned} \tag{6.10}$$

Next, to determine the energy scale $\Lambda_{\rm qcd}$, we calculate the pion decay constant $F_\pi({\rm PS})$ using the obtained mass function Σ and the Pagels-Stokar formula[13] which reads

$$F_\pi({\rm PS})^2 = \frac{N_c}{2\pi^2}\int_0^\infty dx\ \frac{x\Sigma(x)\,(\,\Sigma(x)-x\Sigma'(x)/2\,)}{(\,x+\Sigma(x)^2\,)^2}\ . \tag{6.11}$$

Since we know the Pagels-Stokar formula to agree with the ladder exact result to $10 \sim 20\%$,[6,14] imposing the value $F_\pi({\rm PS}) = 93\sqrt{2}{\rm MeV}$ allows us to fix $\Lambda_{\rm qcd}$. As a consistency check we also calculate the expectation value of the quark bilinear $\langle\bar\psi\psi\rangle_{\rm 1GeV}$,[6] using the formula

$$\langle\bar\psi\psi\rangle_{\rm 1GeV} = -\left(\frac{\alpha(\Lambda)}{\alpha(1{\rm GeV})}\right)^{\frac{9C_2}{11N_c-2N_f}} \frac{N_c}{4\pi^2}\int_0^{\Lambda^2} dx\ \frac{x\Sigma(x)}{x+\Sigma(x)^2} \tag{6.12}$$

with sufficiently large ultraviolet cutoff Λ, where $N_f = 3$ is the number of light flavours. This value (6.12) should be compared with the following "experimental" value given by Gasser and Leutwyler[15]

$$-\langle\bar\psi\psi\rangle_{\rm 1GeV} = (\ 225 \pm 25\ {\rm MeV}\ )^3\ .$$

The results obtained by this procedure are listed in Table.1.

x_0	$\Lambda_{\rm qcd}$	$m_{\rm type\ I}$	$m_{\rm type\ II}$	$(-\langle\bar\psi\psi\rangle_{\rm 1\,GeV})^{1/3}$
1.01	638	489	288	212
1.05	631	482	286	213
1.10	625	474	285	214

Table.1. Parameter values used in our calculation. Units are MeV.

6.3. Bethe-Salpeter equation

In order to solve the BS equation (2.25) numerically, we discretise the variables u and x so that it reduces to a finite dimensional eigenvalue problem, which we solve using a standard linear algebra package. Since logarithmic scale seems natural, we discretise the variables $U \equiv \ln(u/\Lambda_{\rm qcd})$ and $X \equiv \ln(x/\Lambda_{\rm qcd})$ at $N_{\rm BS} = 30$ points evenly spaced in the intervals

$$U \in [\,-10.0,\, 2.5\,]\,, \quad X \in [\,-4.5,\, 4.0\,]\,. \tag{6.13}$$

The integration kernel $K_0(u,x;v,y)$, given explicitly in the appendix, has an integrable logarithmic singularity at $(u,x) = (v,y)$ which we avoide by using the four point average prescription[7]. Calculation of the binding energy E_0 and the corresponding BS amplitude $\chi_0(u,x)$ is straightforward, and we can immediately calculate the decay constant F_B by integrating the real part of $A_0(u,x)$ via eq.(3.3). The number of sites $N_{\rm BS}$ and the support of U and X in eq.(6.13) are large enough to guarantee the discretisation independence of the binding energy and the decay constant to within 1 % — see Table.2. As a consistency check, we also confirm that for the fixed coupling hydrogen atom like case our program gives results which agree well with the formula $E_0 = m\alpha^2/2$ for small α.

type	x_0	$N_{\rm BS}$	$F_B\sqrt{M_B}$	$E_0^{(0)}$	$E_0^{(1)}$
I	1.05	18	3468	935.0	498.3
I	1.05	30	3484	931.7	499.4
I	1.05	34	3486	931.4	499.3

Table.2. $N_{\rm BS}$ independence. $E_0^{(0)}$ and $E_0^{(1)}$ are binding energies of ground state and first excited state, respectively. Units are $({\rm MeV})^{3/2}$ for $F_B\sqrt{M_B}$ and MeV for $E_0^{(0)}$ and $E_0^{(1)}$.

As can be seen in figure 2, the main supports of the integrands of normalisation (2.28) and decay constant (3.3) are included in the ranges given in (6.13).

Unfortunately we can also see that these supports extend far into the infra-red — much further than in the case of the pion[6] — and consequently our results depend on the infra-red behaviour of the coupling. We are therefore forced to include x_0 as a parameter in our model.

Fig.2

Our numerical results for a range of values of x_0 are shown in Table.3. We prefer the value $x_0 = 1.05$ for the simple reason that at this value the shape of the running coupling most resembles the form used in our previous work⋆.[6,7,14] Ideally we would use the excitation energies $E_0^{(0)} - E_0^{(1)}$ to fix the parameter x_0, but at present there is no experimental data for the masses of the excited pseudoscalar or vector B or D mesons.

type	x_0	light quark mass	$F_B\sqrt{M_B}$	$E_0^{(0)}$	$E_0^{(1)}$	$E_0^{(0)} - E_0^{(1)}$
I	1.01	489	2551	1795	1071	724
I	1.05	482	3468	935	498	437
I	1.10	474	4093	666	319	347
II	1.01	288	2052	1558	972	586
II	1.05	286	2738	799	447	352
II	1.10	285	3205	566	279	287

Table.3. x_0 dependence; $N_{\mathrm{BS}} = 18$. Units are MeV except for $F_B\sqrt{M_B}$ which is in $(\mathrm{MeV})^{3/2}$.

The B and D meson decay constants and mass difference using a finer discretisation ($N_{\mathrm{BS}} = 30$) are given in Table.4 where these meson masses are taken from

⋆ Indeed, we have calculated $F_B\sqrt{M_B}$ using the previous running coupling function[6] and confirmed that it produces almost the same result as the present one for the choice $x_0 = 1.05$.

Ref.[16].

type	x_0	mass	$F_{B(5279)}$	$F_{D(1869)}$	$B(1\text{st}) - B(5279)$
I	1.05	482	48.0	80.6	432
II	1.05	286	37.8	63.4	350

Table.4. Results for the meson decay constants and mass difference with $N_{\text{BS}} = 30$ and $x_0 = 1.05$. $B(1\text{st})$ denotes 1st excited state of pseudoscalar B meson. Units are MeV.

As can be seen, our result for F_B is smaller than F_π, and is much smaller than other values obtained using QCD sum rules, potential models and lattice simulations (see the summary in the paper by Rosner[17]) — we note however that one gluon exchange models tend to give small values,[18] although even these are somewhat larger than our results.

6.4. The Isgur-Wise function : $\xi(t)$

After evaluating the BS amplitude of the ground state, we can now apply the formalism of section 4 and calculate the Isgur-Wise function $\xi(t)$ by numerically evaluating the integral eq.(4.7). This integral itself is three dimensional, over u', x' and $\cos\theta$, but as explained in section 4, the calculation of the initial BS amplitudes $A(u,x)$, $B(u,x)$ with complex arguments u and x requires the evaluation of the BS equation (2.25) which involves a further two dimensional integral over v and y. So in fact we are calculating a five dimensional integral, which takes a lot of computer time when the number N_{BS} of points on which the arguments u, x, v and y of the BS amplitudes are discretised becomes as large as 30. In order to save computer time, we evaluate the angle integral over $\cos\theta$ using the Gauss-Legendre formula for numerical integration which is known to give rather precise values with using a relatively small number of data points N_{GL}.

The main experimental interest at the moment is in semi-leptonic decay processes such as $B \to D^{(*)} l\nu$, since these should allow calculation of the Kobayashi-

Maskawa matrix element $V_{\rm cb}$ [5]. For these processes, the invariant mass between l and ν must be positive, and this bounds t from above

$$t \leq \frac{M_B^2 + M_{D^{(*)}}^2}{2M_B M_{D^{(*)}}} \simeq 1.5 - 1.6 \ . \tag{6.14}$$

We have chosen the parameter κ in our running coupling so that we may evaluate the Isgur-Wise function within this region. The result can be seen in figure 3, where we have shown the Isgur-Wise function for the two cases of type I and type II masses. Figure 3 is the main numerical result in this paper.

Fig.3

In figures 4 and 5, we show how the result depends on the choice of the parameter x_0 in the running coupling constant, for both the type I and type II masses. These data are based on the BS solutions on a coarser lattice with $N_{\rm BS} = 18$ points and $\cos\theta$ integration evaluated by $N_{\rm GL} = 10$ Gauss-Legendre formula. We see that the result is almost independent of x_0 for the type I case, whereas there is some dependence for the type II case.

Fig.4

Fig.5

Here a comment may be in order on some 'fluctuations' of our data points. We observe 'large' fluctuations occur for instance, at $t = 1.5 (x_0 = 1.01)$ point in Fig.4, $t = 1.18$(type I) point in Fig.3 and so on. We do not yet understand the precise origin of this phenomenon. But we suspect that some 'coherence' is occurring between the Gauss-Legendre formula and the discretisation for the BS amplitude

data. Indeed, if we increase $N_{\rm GL}$ of the Gauss-Legendre formula, the Isgur-Wise function calculated by using the same BS data becomes more smooth as a whole, *but* the places at which such 'large' fluctuations occur change. Actually, for example, the large deviation at $t = 1.5(x_0 = 1.01)$ in Fig.4 does not appear when $N_{\rm GL} = 6$. (Therefore the seeming 'oscillation' observed in Fig.5 around $t = 1.3 \sim 1.5$ will also be a fake.) If we are allowed to average all the data obtained with various $N_{\rm GL}(\geq 6)$, we obtain very smooth curves which coincide with the cited $N_{\rm GL} = 10$ data with fluctuations smoothed out.

The notable feature of these figures 4 and 5 is that the charge radius of the Isgur-Wise function

$$\rho^2 \equiv -\frac{d\xi(t)}{dt}\Big|_{t=1} \tag{6.15}$$

is as large as 2.0 for type I mass or 1.8 for type II, and is quite insensitive to the choice of x_0. These values of ρ^2 are certainly larger than the usual predictions of $\rho^2 = 1.1 \sim 1.3$.[17,19,20] In our model the slope of the Isgur-Wise function does not decrease to 1.1 until region $t = 1.1 \sim 1.3$. If this feature of our result is correct, it would imply that $V_{\rm cb}$ as extracted from the experimental data[21,22,23,24,25,26] has been underestimated by several percent.

To see this more explicitly, we fit our Isgur-Wise function $\xi(t)$ of figure 3 to the ARGUS data[27] (with setting their parameter $\tau_B = 1.32$ ps) by adjusting $V_{\rm cb}$ so as to minimise $\chi^2 = \sum_n \left(|V_{\rm cb}|\, \xi(t_n) - f_n \right)^2 / \sigma_n^2$, where f_n and σ_n are the experimental data and standard deviations at $t = t_n$. The result is shown in figure 6. We find

$$|V_{\rm cb}| = \begin{cases} 0.0503 & \text{for type I case} \\ 0.0437 & \text{for type II case} \end{cases} , \tag{6.16}$$

and see that the type I case gives a better overall fit to the ARGUS data.

Fig.6

Finally, we use our Isgur-Wise function with type I mass to estimate the ρ^2 parameters in various functional forms of $\xi(t)$ proposed so far.[21,22,24,28,29,30] The functional forms and the corresponding charge radii ρ^2 which give the best fit to our data are given in Table.5, and the resulting curves are shown in figure 7. The model C gives the least χ^2, although the models B and C also give good fits.

model	function form	ρ^2
A	$1-\rho^2(t-1)$	1.31
B	$\frac{2}{t+1}\exp\left[-(2\rho^2-1)\frac{t-1}{t+1}\right]$	2.16
C	$\left(\frac{2}{t+1}\right)^{2\rho^2}$	2.03
D	$\exp[-\rho^2(t-1)]$	1.87

Table.5. Various parametrisations of the Isgur-Wise function. The charge radii are extracted from our $\xi(t)$ with $N_{\mathrm{BS}}=30$, $x_0=1.05$ and type I mass in Fig.3.

Fig.7

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APPENDIX : BS Kernel After Angle Integration

After performing three-dimensional angle integration over $\cos\theta = \mathbf{k}\cdot\mathbf{p}/|\mathbf{k}||\mathbf{p}|$, the zeroth order kernel $K_0(p,k)$ is found to be given by

$$K_0(p,k) = \begin{pmatrix} I_1 - (u-v)^2 I_2 & i(u-v)(y^2 I_2 - I_2^{\mathbf{p}\cdot\mathbf{k}}) \\ -i(u-v)(x^2 I_2 - I_2^{\mathbf{p}\cdot\mathbf{k}}) & I_1^{\mathbf{p}\cdot\mathbf{k}} - (u-v)^2 I_2^{\mathbf{p}\cdot\mathbf{k}} \end{pmatrix} , \tag{A.1}$$

where

$$\begin{aligned} I_1 &\equiv \int_{-1}^{1} d\cos\theta \frac{1}{-(k-p)^2} = \frac{1}{2xy}\ln\left(\frac{(x+y)^2+(u-v)^2}{(x-y)^2+(u-v)^2}\right) \\ I_2 &\equiv \int_{-1}^{1} d\cos\theta \frac{1}{(k-p)^4} = \frac{2}{\big((x+y)^2+(u-v)^2\big)\big((x-y)^2+(u-v)^2\big)} \\ I_1^{\mathbf{p}\cdot\mathbf{k}} &\equiv \int_{-1}^{1} d\cos\theta \frac{\mathbf{k}\cdot\mathbf{p}}{-(k-p)^2} = \frac{x^2+y^2+(u-v)^2}{2} I_1 - 1 \\ I_2^{\mathbf{p}\cdot\mathbf{k}} &\equiv \int_{-1}^{1} d\cos\theta \frac{\mathbf{k}\cdot\mathbf{p}}{(k-p)^4} = \frac{1}{2}\big((x^2+y^2+(u-v)^2) I_2 - I_1\big) . \end{aligned} \tag{A.2}$$

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FIGURE CAPTIONS

Figure 2. Integrands of normalisation (2.28) and F_B (3.3). $N_{\rm BS} = 30$ and $x_0 = 1.05$. The upper 9/10 of the figures are clipped.

Figure 3. Isgur-Wise functions calculated with $N_{\rm BS} = 30$, $N_{\rm GL} = 10$ and $x_0 = 1.05$ for both type I and type II masses.

Figure 4. x_0 dependence of the Isgur-Wise function with type I mass, $N_{\rm BS} = 18$ and $N_{\rm GL} = 10$.

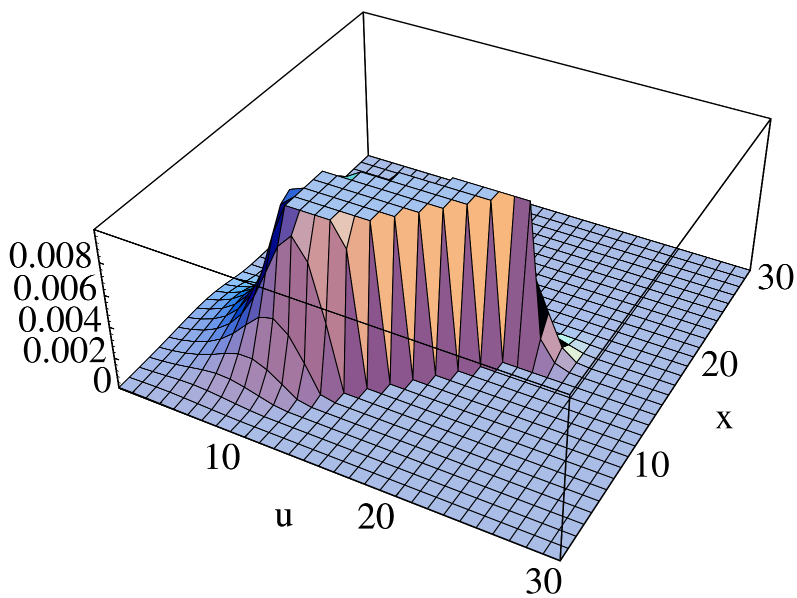
Figure 5. x_0 dependence of the Isgur-Wise function with type II mass,$N_{\rm BS} = 18$ and $N_{\rm GL} = 10$.

Figure 6. $\xi(y) \cdot |V_{\rm cb}|$ versus y. $x_0 = 1.05$ and both type I and II masses are used.

Figure 7. Least squares fits of the four theoretical models A, B, C and D to our numerical data.

Fig.2

Integrand of Normalization



Integrand of Decay Constant

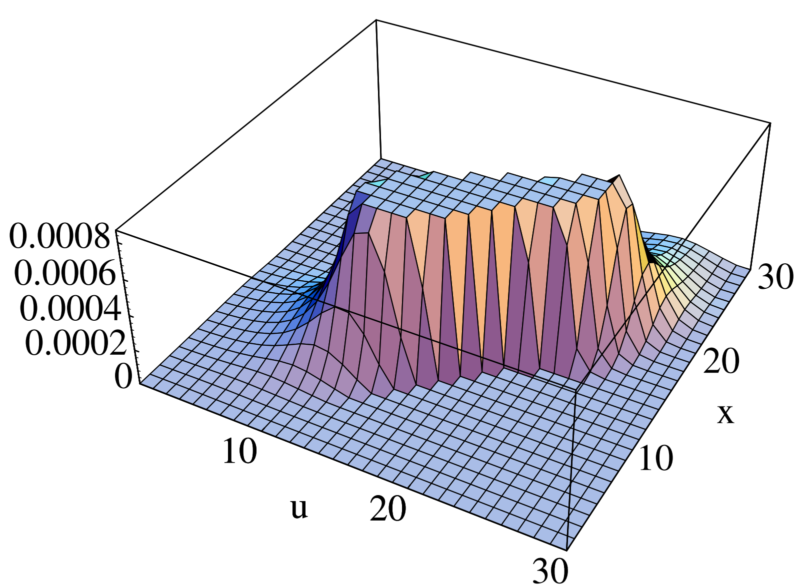


Fig.3

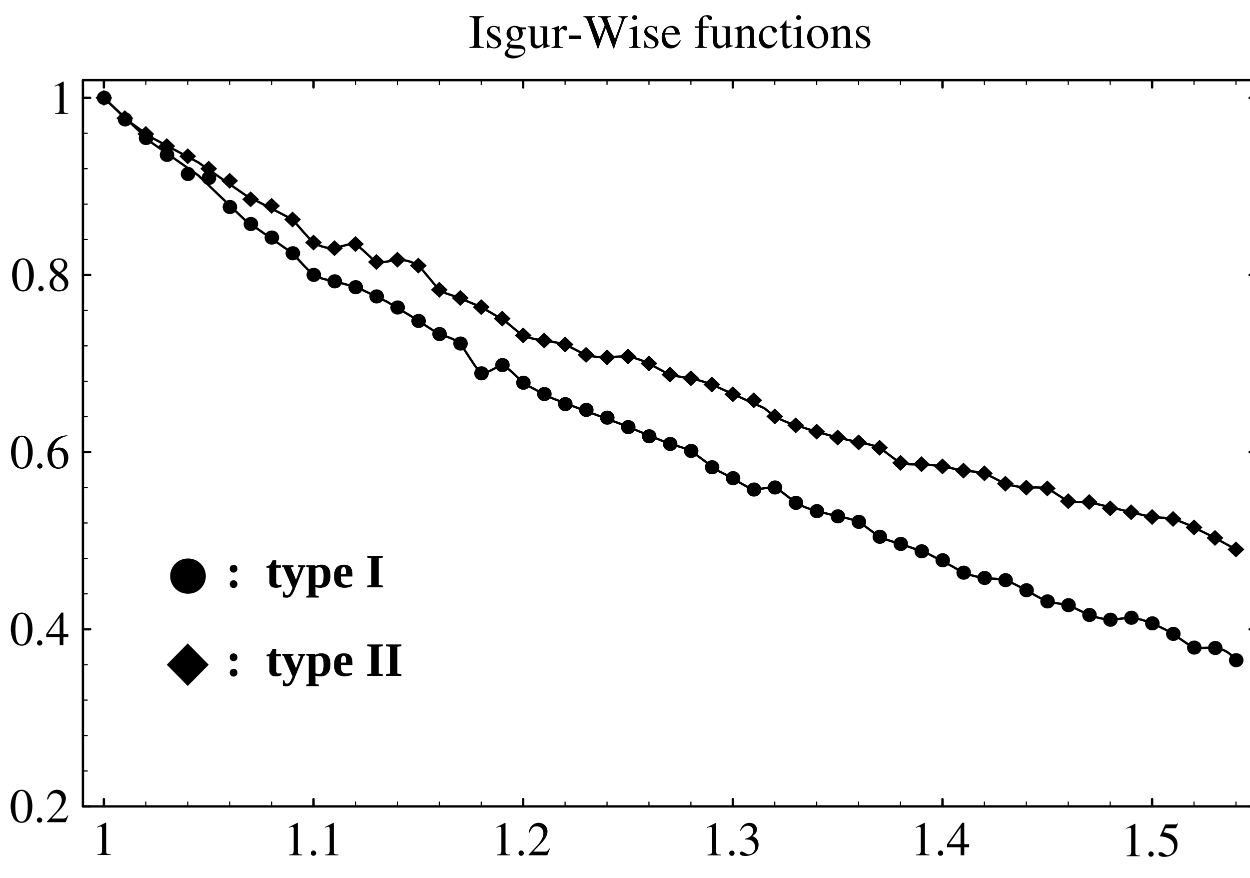
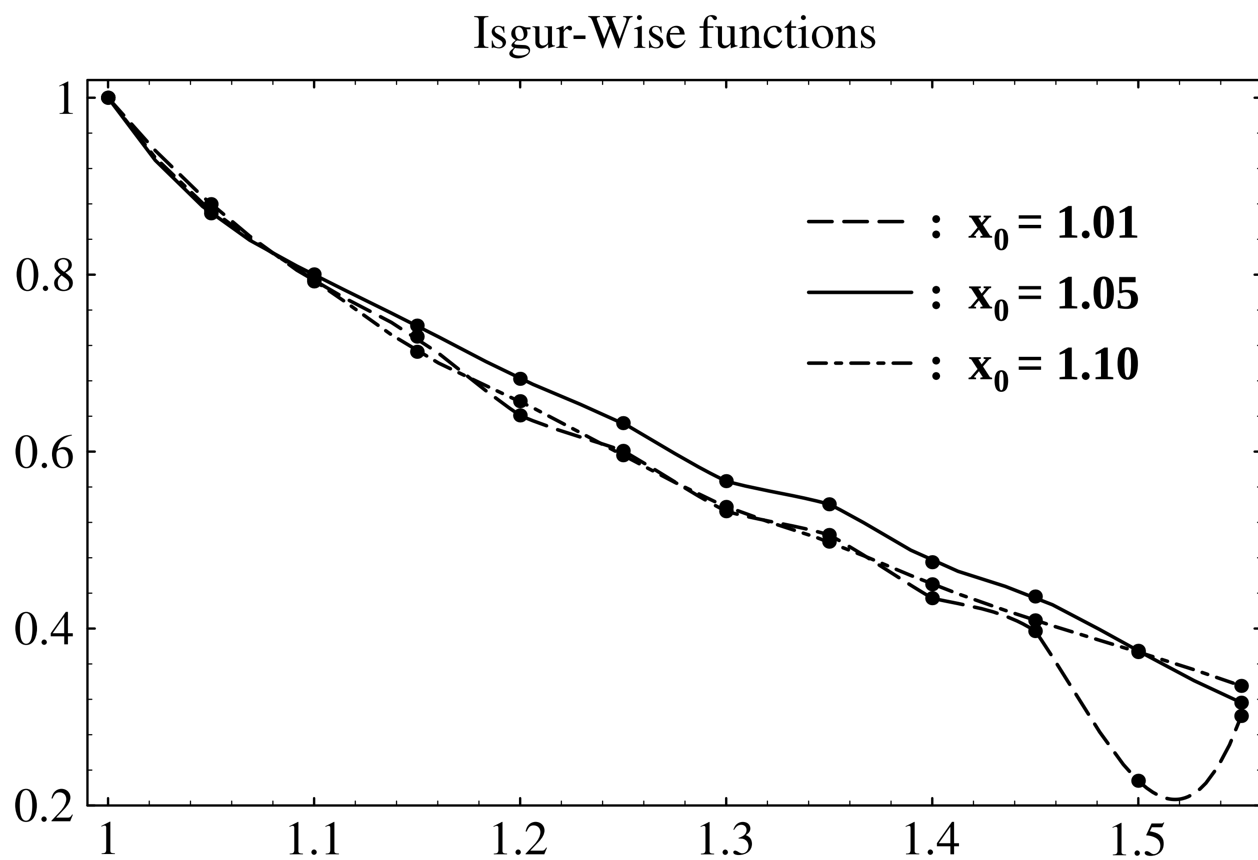


Fig.4


Isgur-Wise functions
: $x_0 = 1.01$
: $x_0 = 1.05$
: $x_0 = 1.10$
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0.8
0.6
0.4
0.2
1
1.1
1.2
1.3
1.4
1.5

Fig.5

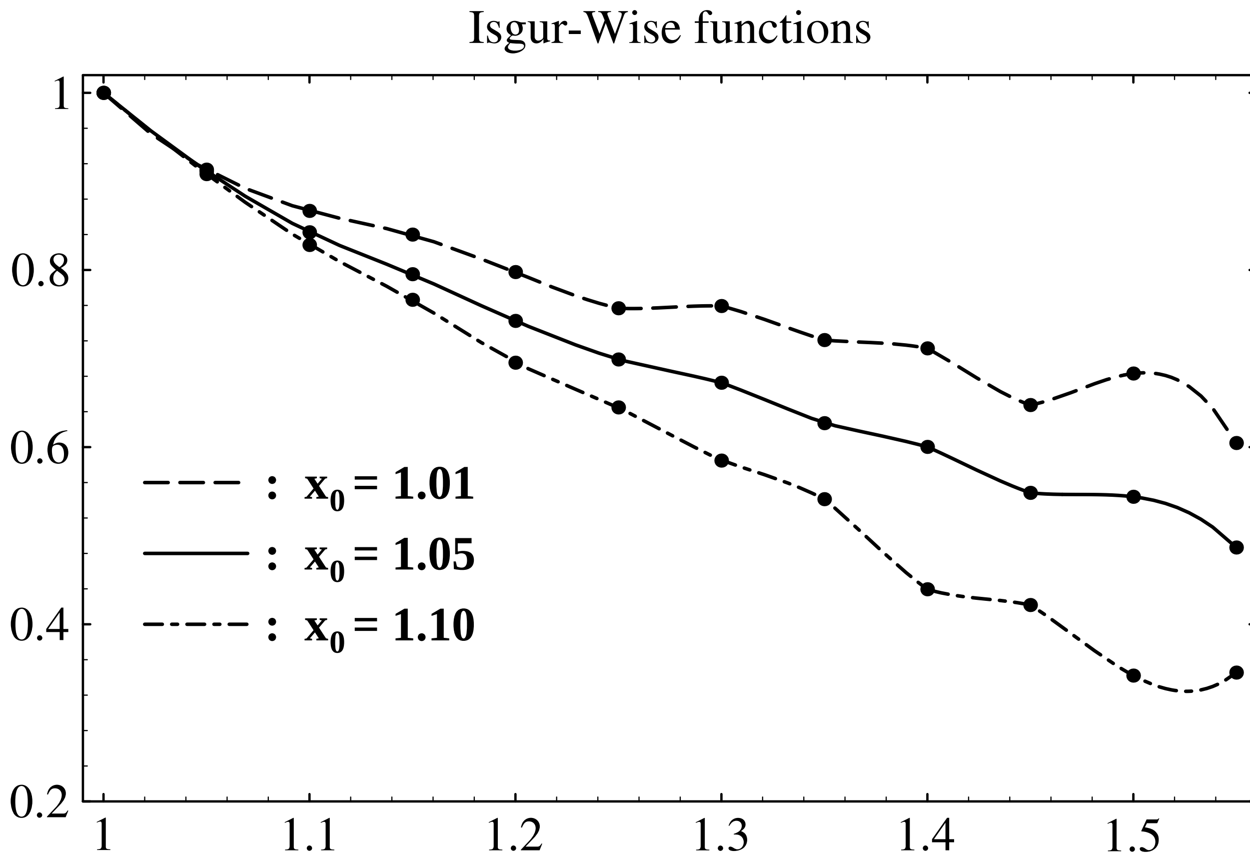
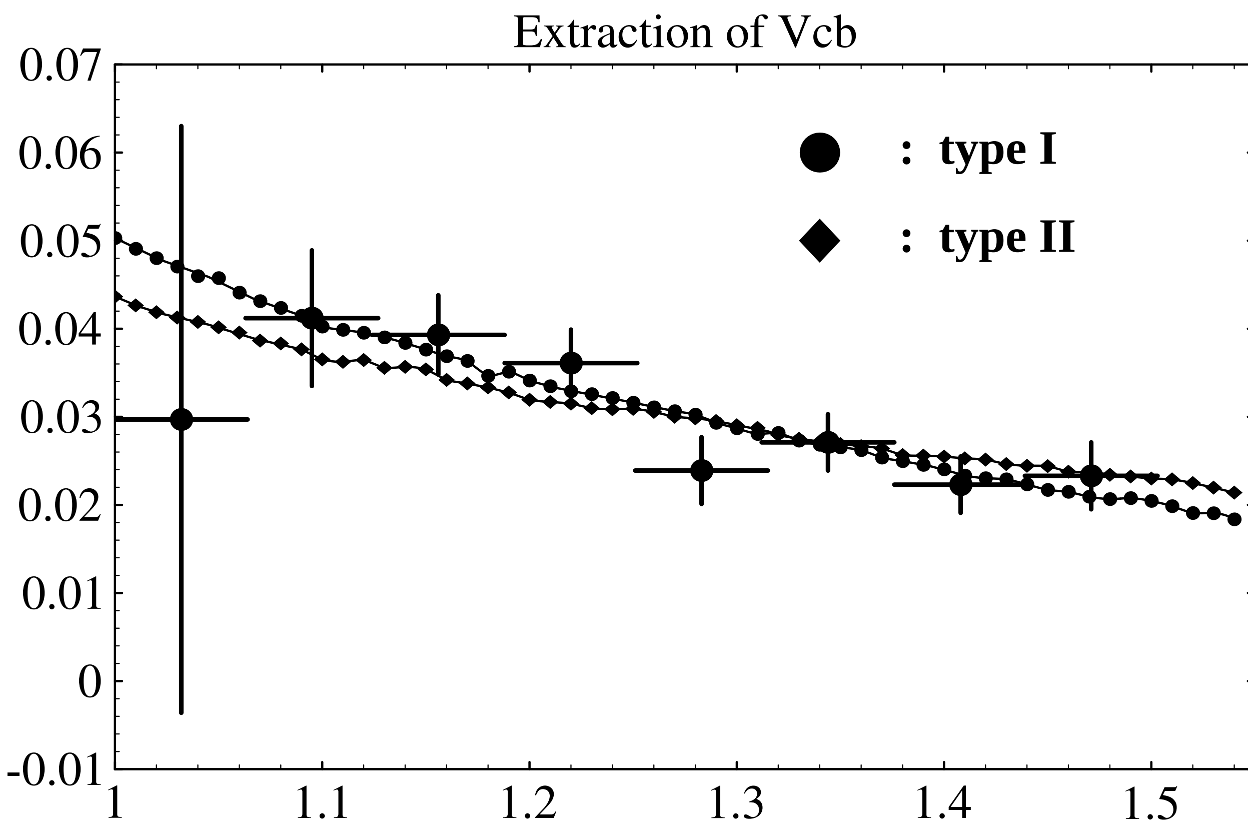

Isgur-Wise functions
1
0.8
0.6
0.4
0.2
1
1.1
1.2
1.3
1.4
1.5
: x0 = 1.01
: x0 = 1.05
: x0 = 1.10

Fig.6


Extraction of Vcb
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0.06
0.05
0.04
0.03
0.02
0.01
0
-0.01
1
1.1
1.2
1.3
1.4
1.5
: type I
: type II

Fig.7